

On Dimensional Reduction of Magical Supergravity Theories

Naoto Kan^{*} and Shun'ya Mizoguchi[†]

^{*†}*SOKENDAI (The Graduate University for Advanced Studies)*

Tsukuba, Ibaraki, 305-0801, Japan and

[†]*Theory Center, Institute of Particle and Nuclear Studies, KEK*

Tsukuba, Ibaraki, 305-0801, Japan

(Dated: May 6, 2016)

Abstract

We prove, by a direct dimensional reduction and an explicit construction of the group manifold, that the nonlinear sigma model of the dimensionally reduced three-dimensional $\mathbb{A} = \mathbb{R}$ magical supergravity is $F_{4(+4)}/(USp(6) \times SU(2))$. This serves as a basis for the solution generating technique in this supergravity as well as allows to give the Lie algebraic characterizations to some of the parameters and functions in the original $D = 5$ Lagrangian. Generalizations to other magical supergravities are also discussed.

PACS numbers: 04.50.-h, 04.65.+e

^{*} E-mail: naotok@post.kek.jp

[†] E-mail: mizoguch@post.kek.jp

I. INTRODUCTION

One of the common features of dimensionally reduced supergravity theories is that they contain a noncompact scalar coset sigma model in their Lagrangians. Perhaps the most famous example is the $E_{7(+7)}/SU(8)$ coset in $D = 4$ $\mathcal{N} = 8$ supergravity [1] obtained by a dimensional reduction of the eleven-dimensionally supergravity [2] to four dimensions. It is always the case that the global symmetry of the nonlinear sigma model is a symmetry of the whole supergravity system including the fermionic sector. A reduction of the eleven-dimensional supergravity to an intermediate dimension from 5 to 10 also yields an E -series symmetry [3, 4], whose discrete subgroup is nowadays understood as a U-duality [5] of M-theory or type-II string theories. It is also known that the symmetry is enhanced to E_8 or much larger (infinite-dimensional) upon reduction to three or lower dimensions [6–9].

The E -series is a token of lower-dimensional M/typeIIA/typeIIB theories upon toroidal compactifications. The D -series, on the other hand, is known to appear as a similar symmetry group of the non-linear sigma model of the dimensionally reduced NS-NS sector supergravity, whose discrete subgroup is a T-duality of the toroidally compactified string theory [12]. It is also very well known that the A -series is a symmetry of dimensionally reduced pure gravity [10, 11]. The B -series may be obtained as a reduction of the NS-NS sector coupled to an odd number of vector fields [12, 13], and $G_{2(+2)}$ has been shown to be the symmetry of the dimensionally reduced $D = 5$ minimal supergravity to three dimensions [14]. So what about the remaining simple Lie algebras?

As for F_4 , many years ago it was anticipated that $F_{4(+4)}/(USp(6) \times SU(2))$ should be the sigma model of the dimensionally reduced $D = 5$ *magical supergravity* of the simplest kind, reduced down to three dimensions [16]^{1 2}. Although the appearance of this particular quaternionic manifold has been justified on various grounds and is now believed to be true, a direct proof by performing a dimensional reduction of the supergravity and comparing to the construction to the coset group manifold seems to have never appeared in print. The aim of this letter is to fill this gap.

The direct proof of the $F_{4(+4)}/(USp(6) \times SU(2))$ coset structure has the following benefits:

¹ Among the C -series, which is also missing in the above description, $Sp(6, \mathbb{R})/U(3)$ ($Sp(6) = C_3$) has also been shown to appear [15] as a scalar coset of the same magical supergravity reduced to *four* dimensions.

² See [17, 18] for the gaugings of the three-dimensional magical supergravities.

(1) The direct dimensional reduction and the explicit construction of the coset sigma model enable us to find the precise relationship between the various components of the five-dimensional supergravity fields and the relevant group elements. This allows us to use the $F_{4(+4)}$ global symmetry to generate a new supergravity solution from some known seed solution. Such a solution-generating technique utilizing the three- or four-dimensional global symmetry has been very powerful in deriving, for instance, the five-dimensional black hole solutions in five-dimensional minimal supergravity [19].

(2) By the above relationship between the supergravity fields and the group manifold one can also give group theoretical characterizations to some of the parameters and functions in the original magical supergravity Lagrangians. For example, as we show below, the FFA coupling constants C_{IJK} are identified as the structure constants of the commutation relations between generators both belonging to one of the “Jordan pair” in the decomposition [22] of the quasi-conformal algebra of the relevant Jordan algebra. We will also find a Lie algebraic characterization of the functions of the scalars \mathring{a}^{IJ} and \mathring{a}_{IJ} .

In fact, the procedure of the dimensional reduction itself is common to all the magical supergravity theories; the only difference is the range of the values of the indices of the vector and scalar fields. Although the three-dimensional duality Lie algebras also allow a common decomposition in terms of the relevant Jordan algebras [15, 16, 20–22], in this letter we will work out in particular the $F_{4(+4)}$ case in detail. We expect, however, a similar identification or a characterization of the coupling constants and scalar metric functions may be done in other magical supergravities.

II. DIMENSIONAL REDUCTION OF $D = 5$ MAGICAL SUPERGRAVITY

The magical supergravities are $D = 5$ $\mathcal{N} = 2$ Einstein-Maxwell supergravities whose scalars of the vector multiplets constitute a coset sigma model with a symmetry group being a *simple* Lie group [15]. There exist four such theories, each of which is associated with one of the four division algebras $\mathbb{A} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ and a rank-3 Jordan algebra $J_3^{\mathbb{A}}$ associated with it. One of the characteristic features of these theories is that their five-dimensional Lagrangians as well as their dimensional reductions to four and three dimensions universally contain scalar sigma models of the forms [15, 22]:

$$\frac{\text{Str}_0(J_3^{\mathbb{A}})}{\text{Aut}(J_3^{\mathbb{A}})} \quad (D = 5), \quad \frac{\text{Mö}(J_3^{\mathbb{A}})}{\widetilde{\text{Str}}_0(J_3^{\mathbb{A}}) \times U(1)} \quad (D = 4), \quad \frac{\text{qConf}(J_3^{\mathbb{A}})}{\widetilde{\text{Mö}}(J_3^{\mathbb{A}}) \times SU(2)} \quad (D = 3), \quad (1)$$

where $\text{Aut}(\mathbb{J}_3^{\mathbb{A}})$, $\text{Str}_0(\mathbb{J}_3^{\mathbb{A}})$, $\text{Mö}(\mathbb{J}_3^{\mathbb{A}})$ and $\text{qConf}(\mathbb{J}_3^{\mathbb{A}})$ are respectively the automorphism group, the reduced structure group, the superstructure group and the quasi-conformal group of the Jordan algebra $\mathbb{J}_3^{\mathbb{A}}$. $\widetilde{}$ denotes the corresponding compact form. These supergravity theories have been dubbed “magical” [16] because these groups are precisely the elements of the “magic square” (see [16] and references therein), each Lie algebra $\mathcal{L}_{\mathbb{A},\mathbb{A}'}$ of which allows the decomposition

$$\mathcal{L}_{\mathbb{A},\mathbb{A}'} = \mathbb{D}_{\mathbb{A}} \oplus \mathbb{D}_{\mathbb{J}_3^{\mathbb{A}'}} \oplus (\mathbb{A}_0 \times (\mathbb{J}_3^{\mathbb{A}'}))_0, \quad (2)$$

where $\mathbb{A}' = \mathbb{R}, \mathbb{C}, \mathbb{H}$ and \mathbb{O} corresponds to Aut , Str_0 , Mö and qConf , respectively. Here $\mathbb{D}_{\mathbb{A}}$ and $\mathbb{D}_{\mathbb{J}_3^{\mathbb{A}'}}$ are the generators of the automorphisms of \mathbb{A} and $\mathbb{J}_3^{\mathbb{A}'}$, and \mathbb{A}_0 and $(\mathbb{J}_3^{\mathbb{A}'})_0$ are the traceless generators.

The magical supergravity corresponding to the division algebra \mathbb{A} has $n = 3(1 + \dim \mathbb{A}) - 1$ vector multiplets. Keeping only the bosonic terms, the Lagrangian is given by

$$\begin{aligned} \mathcal{L} = & \frac{1}{2}E^{(5)}R^{(5)} - \frac{1}{4}E^{(5)}\overset{\circ}{a}_{IJ}F_{MN}^IF_{MN}^{JMN} - \frac{1}{2}E^{(5)}s_{xy}(\partial_M\phi^x)(\partial^M\phi^y) \\ & + \frac{1}{6\sqrt{6}}C_{IJK}\epsilon^{MNPQR}F_{MN}^IF_{PQ}^JA_R^K, \end{aligned} \quad (3)$$

where $E^{(5)}$ is the determinant of the fünfbein, and $R^{(5)}$ is the scalar curvature in $D = 5$. $\overset{\circ}{a}_{IJ}$ and s_{xy} are functions of scalar fields ϕ^x which come from the vector multiplets and satisfy $\overset{\circ}{a}_{IJ} = \overset{\circ}{a}_{JI}$ and $s_{xy} = s_{yx}$, respectively. In particular, s_{xy} is the metric of n -dimensional Riemannian space \mathcal{M} which is parametrized by the scalar fields ϕ^x , where x, y, \dots take $1, 2, \dots, n$. F_{MN}^I is the Maxwell field strength $2\partial_{[\mu}A_{\nu]}^I$. C_{IJK} is a constant and symmetric in all indices. M, N, \dots are the five-dimensional curved indices. There are $n + 1$ vector fields A_μ^I because the graviton multiplet has a single vector field, so that $I, J, \dots = 1, 2, \dots, n + 1$.

To reduce the dimensions to $D = 3$, we set the fünfbein and its inverse as

$$E^{(5)}{}_M{}^A = \begin{pmatrix} e^{-1}E_\mu{}^\alpha & B_\mu{}^m e_m{}^a \\ 0 & e_m{}^a \end{pmatrix}, \quad E^{(5)}{}_A{}^M = \begin{pmatrix} eE_\alpha{}^\mu & -eE_\alpha{}^\mu B_\mu{}^m \\ 0 & e_a{}^m \end{pmatrix}, \quad (4)$$

where A, B, \dots are the five-dimensional flat indices, μ, ν, \dots and α, β, \dots are the three-dimensional curved and flat indices, m, n, \dots and a, b, \dots are the compact two-dimensional curved and flat indices, respectively. Then we get the reduced Lagrangian

$$\begin{aligned} \mathcal{L} = & \frac{1}{2}ER - \frac{1}{8}Ee^2g_{mn}B_\mu{}^m B^{\mu\nu} + \frac{1}{8}E\partial_\mu g^{mn}\partial^\mu g_{mn} - \frac{1}{2}Ee^{-2}\partial_\mu e\partial^\mu e - \frac{1}{2}Es_{xy}(\partial_\mu\phi^x)(\partial^\mu\phi^y) \\ & - \frac{1}{2}E\overset{\circ}{a}_{IJ}g^{mn}\partial_\mu A_m^I\partial^\mu A_n^J - \frac{1}{4}Ee^2\overset{\circ}{a}_{IJ}F_{\mu\nu}^{(3)I}F^{(3)J\mu\nu} + \frac{1}{\sqrt{6}}C_{IJK}\epsilon^{\mu\nu\rho}\epsilon^{mn}F_{\mu\nu}^I\partial_\rho A_m^JA_n^K, \end{aligned} \quad (5)$$

where $B_{\mu\nu}^m = 2\partial_{[\mu}B_{\nu]}^m$. We define $F_{\mu\nu}^{(3)I} \equiv F_{\mu\nu}'^I + B_{\mu\nu}^m A_m^I$, where $F_{\mu\nu}'^I = 2\partial_{[\mu}A_{\nu]}'^I$ is the field strength of the Kaluza-Klein invariant vector field $A_\mu'^I = A_\mu^I - B_\mu^m A_m^I$.

To dualize $A_\mu'^I$ and B_μ^m fields, we introduce Lagrange multipliers

$$\begin{aligned}\mathcal{L}_{\text{Lag.mult.}} &= \epsilon^{\mu\nu\rho}\varphi_I\partial_\mu F_{\nu\rho}'^I + \frac{1}{2}\epsilon^{\mu\nu\rho}\psi_m\partial_\mu B_{\nu\rho}^m \\ &\stackrel{\text{P.I.}}{=} -\epsilon^{\mu\nu\rho}F_{\mu\nu}^{(3)I}\partial_\rho\varphi_I - \frac{1}{2}\epsilon^{\mu\nu\rho}B_{\mu\nu}^m(\partial_\rho\psi_m + \partial_\rho A_m^I\varphi_I - A_m^I\partial_\rho\varphi_I).\end{aligned}\quad (6)$$

Using the equations of motion for $F_{\mu\nu}^{(3)I}$ and $B_{\mu\nu}^m$, we obtain the dualized Lagrangian $\tilde{\mathcal{L}} \equiv \mathcal{L} + \mathcal{L}_{\text{Lag.mult.}}$:

$$\begin{aligned}\tilde{\mathcal{L}} &= \frac{1}{2}ER + \frac{1}{8}E\partial_\mu g^{mn}\partial^\mu g_{mn} - \frac{1}{2}Ee^{-2}\partial_\mu e\partial^\mu e - \frac{1}{2}Es_{xy}(\partial_\mu\phi^x)(\partial^\mu\phi^y) - \frac{1}{2}E\hat{a}_{IJ}g^{mn}\partial_\mu A_m^I\partial^\mu A_n^J \\ &\quad - 2Ee^{-2}\hat{a}^{II'}\left(\frac{1}{\sqrt{6}}C_{IJK}\epsilon^{mn}\partial_\mu A_m^J A_n^K - \partial_\mu\varphi_I\right)\left(\frac{1}{\sqrt{6}}C_{I'J'K'}\epsilon^{m'n'}\partial^\mu A_{m'}^{J'} A_{n'}^{K'} - \partial^\mu\varphi_{I'}\right) \\ &\quad - Ee^{-2}g^{mn}\left(\frac{2}{3\sqrt{6}}C_{IJK}\epsilon^{pq}\partial_\mu A_p^I A_q^J A_m^K + \partial_\mu\psi_m + \partial_\mu A_m^I\varphi_I - A_m^I\partial_\mu\varphi_I\right) \\ &\quad \times \left(\frac{2}{3\sqrt{6}}C_{I'J'K'}\epsilon^{p'q'}\partial^\mu A_{p'}^{I'} A_{q'}^{J'} A_n^{K'} + \partial^\mu\psi_n + \partial^\mu A_n^{I'}\varphi_{I'} - A_n^{I'}\partial^\mu\varphi_{I'}\right).\end{aligned}\quad (7)$$

III. $F_{4(+4)}/(USp(6) \times SU(2))$ SIGMA MODEL: THE EXPLICIT PROOF

In this section we prove that, if $\mathbb{A} = \mathbb{R}$ ($n = 5$), the sigma model part of the reduced Lagrangian (7) constitutes the $F_{4(+4)}/(USp(6) \times SU(2))$ sigma model by an explicit construction.

The real form $F_{4(+4)}$ of the exceptional Lie algebra F_4 is decomposed into a sum of representations of the Lie algebra of a maximal subgroup $SL(3, \mathbb{R}) \times SL(3, \mathbb{R})$ as

$$\mathbf{52} = (\mathbf{8}, \mathbf{1}) \oplus (\mathbf{3}, \bar{\mathbf{6}}) \oplus (\bar{\mathbf{3}}, \mathbf{6}) \oplus (\mathbf{1}, \mathbf{8}). \quad (8)$$

In spite of the notation, they are represented by real matrices. Later we will identify the first $SL(3, \mathbb{R})$ as the global symmetry group arising from the reduction of the gravity sector from five to three dimensions, and the second one as the numerator group of the coset sigma-model scalars already existing in five dimensions. To distinguish them we call the first simply $SL(3, \mathbb{R})$ while the second $\widetilde{SL}(3, \mathbb{R})$.

Let \hat{E}_j^i ($i, j = 1, 2, 3$) be generators of the $SL(3, \mathbb{R})$ algebra with a constraint $\hat{E}_1^1 + \hat{E}_2^2 + \hat{E}_3^3 = 0$. Similarly let $\hat{\tilde{E}}_{\tilde{b}}^{\tilde{a}}$ $\tilde{a}, \tilde{b} = 1, 2, 3$ be generators of $\widetilde{SL}(3, \mathbb{R})$ with $\hat{\tilde{E}}_1^1 + \hat{\tilde{E}}_2^2 + \hat{\tilde{E}}_3^3 = 0$.

Their commutations relations are

$$[\hat{E}_j^i, \hat{E}_l^k] = \delta_j^k \hat{E}_l^i - \delta_l^i \hat{E}_j^k, \quad (9)$$

$$[\hat{\tilde{E}}_{\tilde{b}}^{\tilde{a}}, \hat{\tilde{E}}_{\tilde{d}}^{\tilde{c}}] = \delta_{\tilde{b}}^{\tilde{c}} \hat{\tilde{E}}_{\tilde{d}}^{\tilde{a}} - \delta_{\tilde{d}}^{\tilde{a}} \hat{\tilde{E}}_{\tilde{b}}^{\tilde{c}}, \quad (10)$$

$$[\hat{E}_j^i, \hat{\tilde{E}}_{\tilde{d}}^{\tilde{c}}] = 0. \quad (11)$$

We also introduce additional generators E_i^I, E_I^{*i} ($i = 1, 2, 3, I = 1, \dots, 6$) transforming respectively as $(\mathbf{3}, \bar{\mathbf{6}}), (\bar{\mathbf{3}}, \mathbf{6})$ under $SL(3, \mathbb{R}) \oplus \widetilde{SL}(3, \mathbb{R})$:

$$[\hat{E}_j^i, E_I^{*k}] = \delta_j^k E_I^{*i}, \quad (12)$$

$$[\hat{E}_j^i, E_k^I] = -\delta_k^i E_j^I, \quad (13)$$

$$[\hat{\tilde{E}}_{\tilde{b}}^{\tilde{a}}, E_I^{*k}] = \bar{T}_{\tilde{b}I}^{\tilde{a}J} E_J^{*i}, \quad (14)$$

$$[\hat{\tilde{E}}_{\tilde{b}}^{\tilde{a}}, E_k^I] = T_{\tilde{b}}^{\tilde{a}I} E_i^J. \quad (15)$$

$\bar{T}_{\tilde{b}I}^{\tilde{a}J}$ and $T_{\tilde{b}}^{\tilde{a}I}$ are respectively the $\bar{\mathbf{6}}$ and $\mathbf{6}$ representation matrices of $\widetilde{SL}(3, \mathbb{R})$. In fact, in the present choice of the basis of the generators the structure constants satisfy

$$\bar{T}_{\tilde{b}I}^{\tilde{a}A} = -T_{\tilde{b}}^{\tilde{a}A} E_I^A. \quad (16)$$

Finally we set the commutation relations among two of these generators as

$$[E_i^I, E_J^{*j}] = -4\delta_J^I \hat{E}_i^j + \delta_i^j D_{J\tilde{a}}^I \tilde{b} \hat{\tilde{E}}_{\tilde{b}}^{\tilde{a}}, \quad (17)$$

$$[E_i^I, E_j^J] = +C^{IJK} \epsilon_{ijk} E_K^{*k}, \quad (18)$$

$$[E_I^{*i}, E_J^{*j}] = -C_{IJK} \epsilon^{ijk} E_k^K, \quad (19)$$

where $C_{IJK} = C^{IJK}$ are symmetric with respect to any permutation of indices, and

$$D_{J\tilde{a}}^I \tilde{b} = D_{I\tilde{b}}^J \tilde{a}. \quad (20)$$

Their actual values in the present basis are

$$\begin{aligned} C^{123} &= \sqrt{2}, \\ C^{456} &= +2, \\ C^{114} &= C^{225} = C^{336} = -2, \end{aligned} \quad (21)$$

and

$$\begin{aligned}
D_{22}^{1\ 1} &= D_{31}^{1\ 3} = +2, \\
D_{33}^{2\ 2} &= -2, \\
D_{63}^{1\ 2} &= D_{52}^{1\ 3} = +2\sqrt{2}, \\
D_{41}^{3\ 2} &= D_{51}^{3\ 2} = D_{41}^{2\ 3} = D_{63}^{2\ 1} = -2\sqrt{2}, \\
D_{11}^{1\ 1} &= D_{22}^{2\ 2} = D_{33}^{3\ 3} = +2, \\
D_{41}^{4\ 1} &= D_{52}^{5\ 2} = D_{63}^{6\ 3} = +4,
\end{aligned} \tag{22}$$

otherwise 0. One may verify that the commutations relations (9)-(19) close and generate the whole $F_{4(+4)}$ Lie algebra.

In fact, these commutations relations are derived from those among generators of a more tractable realization of $F_{4(+4)}$ in terms of the decomposition into representations of another maximal subalgebra $O(4, 5)$:

$$\mathbf{52} = \mathbf{36} \oplus \mathbf{16}, \tag{23}$$

where $\mathbf{36}$ is the adjoint representation of $O(4, 5)$ and $\mathbf{16}$ is the Majorana spinor representation. They are further decomposed into representations of $O(4, 4)$ as

$$\mathbf{52} = \mathbf{28} \oplus \mathbf{8_v} \oplus \mathbf{8_s} \oplus \mathbf{8_c}, \tag{24}$$

which shows the hidden triality in $F_{4(+4)}$. The commutation relations among generators are

$$[X^{ab}, X^{cd}] = \eta^{bc}X^{ad} - \eta^{ac}X^{bd} - \eta^{bd}X^{ac} + \eta^{ad}X^{bc}, \tag{25}$$

$$[X^{ab}, v^c] = \eta^{bc}v^a - \eta^{ac}v^b, \tag{26}$$

$$[v^a, v^b] = -X^{ab}, \tag{27}$$

$$[X^{ab}, s^\alpha] = -\frac{1}{2}(\bar{\gamma}^{[a}\gamma^{b]})^\alpha_\beta s^\beta, \tag{28}$$

$$[X^{ab}, c_\alpha] = -\frac{1}{2}(\bar{\gamma}^{[a}\gamma^{b]})_\alpha^\beta c_\beta, \tag{29}$$

$$[v^a, s^\alpha] = +\frac{1}{2}(\bar{\gamma}^a)^{\alpha\beta} c_\beta, \tag{30}$$

$$[v^a, c_\alpha] = -\frac{1}{2}(\gamma^a)_{\alpha\beta} s^\beta, \tag{31}$$

$$[s^\alpha, s^\beta] = -\frac{1}{2}(\bar{\gamma}_a\gamma_b C)^{\alpha\beta} X^{ab}, \tag{32}$$

$$[c_\alpha, c_\beta] = +\frac{1}{2}(\gamma_a\bar{\gamma}_b C)_{\alpha\beta} X^{ab}, \tag{33}$$

$$[s^\alpha, c_\beta] = +(\gamma_a C)_\beta^\alpha v^a, \tag{34}$$

where $X^{ab} = -X^{ba} \in \mathbf{28}$ ($a, b = 1, \dots, 8$), $v^a \in \mathbf{8_v}$ ($a = 1, \dots, 8$), $s^\alpha \in \mathbf{8_s}$ ($\alpha = 1, \dots, 8$), and $c_\alpha \in \mathbf{8_c}$ ($\alpha = 1, \dots, 8$). Here the conventions are $\eta^{ab} = \text{diag}(-1, -1, -1, -1, 1, 1, 1, 1)$, and $\gamma^a, \bar{\gamma}^a$ ($a = 1, \dots, 8$) are off-diagonal blocks of $O(4, 4)$ gamma matrices in the Majorana-Weyl representation:

$$\Gamma^a = \begin{pmatrix} & \bar{\gamma}^a \\ \gamma^a & \end{pmatrix}. \quad (35)$$

C is the charge conjugation matrix satisfying

$$C\gamma_a^T = -\gamma_a C, \quad (36)$$

$$C\bar{\gamma}_a^T = -\bar{\gamma}_a C. \quad (37)$$

The generators $\hat{E}_j^i \in (\mathbf{8}, \mathbf{1})$ ($i, j = 1, 2, 3$), $\hat{E}_{\tilde{b}}^{\tilde{a}} \in (\mathbf{1}, \mathbf{8})$ ($\tilde{a}, \tilde{b} = 1, 2, 3$), $E_i^I \in (\bar{\mathbf{3}}, \mathbf{6})$ and $E_I^{*i} \in (\mathbf{3}, \bar{\mathbf{6}})$ ($i = 1, 2, 3$, $I = 1, \dots, 6$) in the $SL(3, \mathbb{R}) \times \widetilde{SL}(3, \mathbb{R})$ decomposition can be found as follows:

- One can take \hat{E}_j^i 's as the standard $SL(3, \mathbb{R})$ generators in the $O(3, 3)$ subalgebra of $O(4, 4)$.
- In the remaining generators of $O(4, 4)$ one can find three pairs of $\mathbf{3}$ and $\bar{\mathbf{3}}$ of $SL(3, \mathbb{R})$.
- Also in each of v^a , s^α and c_α one can find a single pair, in total another three pairs, of $\mathbf{3}$ and $\bar{\mathbf{3}}$ of $SL(3, \mathbb{R})$.
- The remaining eight generators that do not belong to any of the above turn out to generate another $SL(3, \mathbb{R})$ algebra, $\widetilde{SL}(3, \mathbb{R})$.
- Finally, one can verify that these six pairs of $\mathbf{3}$ and $\bar{\mathbf{3}}$ respectively transform as $\bar{\mathbf{6}}$ and $\mathbf{6}$ under $\widetilde{SL}(3, \mathbb{R})$.

In terms of the $SL(3, \mathbb{R}) \times \widetilde{SL}(3, \mathbb{R})$ decomposition, the whole $F_{4(+4)}$ generators are classified into \mathbf{H} and \mathbf{K} , of which $F_{4(+4)}$ is a direct sum:

$$F_{4(+4)} = \mathbf{H} \oplus \mathbf{K}. \quad (38)$$

\mathbf{H} consists of “compact” generators:

$$\begin{aligned} \mathbf{H} = & (\oplus_{i,j=1,2,3} \mathbb{R}(\hat{E}_j^i - \hat{E}_i^j)) \oplus (\oplus_{\tilde{a},\tilde{b}=1,2,3} \mathbb{R}(\hat{E}_{\tilde{b}}^{\tilde{a}} - \hat{E}_{\tilde{a}}^{\tilde{b}})) \\ & \oplus (\oplus_{i=1,2,3; I=1,\dots,6} \mathbb{R}(E_i^I - E_I^{*i})). \end{aligned} \quad (39)$$

The Killing bilinear form on \mathbf{H} is negative definite. It turns out that the independent $3 + 3 + 18 = 24$ generators of \mathbf{H} generate $USp(6) \oplus SU(2)$. The generators of this factorized $SU(2)$ are

$$H_i = \frac{1}{2} \left(\hat{E}_{i+2}^{i+1} - \hat{E}_{i+1}^{i+2} \right) + \frac{1}{4} \left(E_i^4 - E_4^{*i} + E_i^5 - E_5^{*i} + E_i^6 - E_6^{*i} \right) \quad (40)$$

($i = 1, 2, 3$), where the indices of \hat{E} are defined modulo 3. H_i 's satisfy the $SU(2)$ commutation relations

$$[H_i, H_j] = -2\epsilon_{ijk}H_k. \quad (41)$$

In fact, this $SU(2)$ is one of the irreducible $SU(2)$ subalgebra of $O(4) = SU(2) \oplus SU(2)$, which itself is an irreducible one of the maximal compact subalgebra $O(4) \oplus O(4)$ of $O(4, 4)$. Thus they trivially commute with other compact generators contained in $O(4, 5) = O(4, 4) \oplus \oplus_{a=1, \dots, 8} \mathbb{R}v_a$. It can also be verified that they also commute with compact generators made out of s^α 's and c_α 's. The remaining orthogonal compliment in \mathbf{H} consisting of 21 generators generates $USp(6)$.

On the other hand, \mathbf{K} is spanned by all the “noncompact” generators:

$$\begin{aligned} \mathbf{K} = & (\oplus_{i,j=1,2,3} \mathbb{R}(\hat{E}_j^i + \hat{E}_i^j)) \oplus (\oplus_{\tilde{a}, \tilde{b}=1,2,3} \mathbb{R}(\hat{\tilde{E}}_{\tilde{b}}^{\tilde{a}} + \hat{\tilde{E}}_{\tilde{a}}^{\tilde{b}})) \\ & \oplus (\oplus_{i=1,2,3; I=1, \dots, 6} \mathbb{R}(E_i^I + E_I^{*i})). \end{aligned} \quad (42)$$

The $52 - 28 = 24$ generators of \mathbf{K} parametrize the “physical” degrees of freedom of the $F_{4(+4)}/(USp(6) \times SU(2))$ nonlinear sigma model.

$F_{4(+4)}/(USp(6) \times SU(2))$ is a symmetric space for which we denote the Cartan involution as τ :

$$[\mathbf{H}, \mathbf{H}] \subset \mathbf{H},$$

$$[\mathbf{K}, \mathbf{K}] \subset \mathbf{H}, \quad (43)$$

$$[\mathbf{H}, \mathbf{K}] \subset \mathbf{K}, \quad (44)$$

$$\tau(\mathbf{H}) = -\mathbf{H}, \quad \tau(\mathbf{K}) = +\mathbf{K}. \quad (45)$$

As usual, to construct a coset nonlinear sigma model, we define some group element \mathcal{V} and consider

$$\mathcal{M} \equiv \tau(\mathcal{V}^{-1})\mathcal{V}. \quad (46)$$

Then the Lagrangian is given, up to a constant, by

$$-\frac{1}{4}E^{(3)}\text{Tr}\partial_\mu\mathcal{M}^{-1}\partial^\mu\mathcal{M} = E^{(3)}\text{Tr}\left(\frac{1}{2}(\partial_\mu\mathcal{V}\mathcal{V}^{-1} + \tau(\partial_\mu\mathcal{V}\mathcal{V}^{-1}))\right)^2. \quad (47)$$

In order to reproduce the dimensionally reduced Lagrangian (7) of the magical supergravity, we take ³

$$\mathcal{V} = \mathcal{V}_-\mathcal{V}_+, \quad (48)$$

$$\mathcal{V}_+ = \mathcal{V}_+^{grav.} + \mathcal{V}_+^{scalar}, \quad (49)$$

$$\begin{aligned} \mathcal{V}_+^{grav.} &= \exp\left(\log e_1^1 \hat{E}_1^1 + \log e_2^2 \hat{E}_2^2 + \log e \hat{E}_3^3\right) \\ &\quad \cdot \exp\left(-e_1^2 e_2^2 \hat{E}_2^1\right) \exp\left(\psi_1 \hat{E}_3^1 + \psi_2 \hat{E}_3^2\right), \end{aligned} \quad (50)$$

$$\mathcal{V}_+^{scalar} = \exp\left((\log \tilde{\mathbf{e}}^{-1})_{\tilde{a}}^{\tilde{i}} \hat{E}_{\tilde{i}}^{\tilde{a}}\right) \quad (\tilde{a}, \tilde{i} = 1, 2, 3), \quad (51)$$

where we have taken the zweibein for the reduced dimensions to be in the upper-triangular form

$$e_{i'}^{a'} = \begin{pmatrix} e_1^1 & e_1^2 \\ 0 & e_2^2 \end{pmatrix} \quad (52)$$

so that

$$e = \det e_{i'}^{a'} = (e_1^1 e_2^2)^{-1}, \quad (53)$$

and

$$\tilde{\mathbf{e}}^{-1} = \begin{pmatrix} s_{11} & s_{12} & s_{13} \\ 0 & s_{22} & s_{23} \\ 0 & 0 & (s_{11}s_{22})^{-1} \end{pmatrix}^{-1}. \quad (54)$$

For \mathcal{V}_- we take

$$\mathcal{V}_- = \exp\left(A_{i'}^I E_I^{*i'} + \varphi_I E_3^I\right) \quad (i' = 1, 2; I = 1, \dots, 6). \quad (55)$$

Then a straightforward calculation yields

$$\partial_\mu\mathcal{V}\mathcal{V}^{-1} = \partial_\mu\mathcal{V}_+\mathcal{V}_+^{-1} + \mathcal{V}_+(\partial_\mu\mathcal{V}_-\mathcal{V}_-^{-1})\mathcal{V}_+^{-1}, \quad (56)$$

$$\begin{aligned} \partial_\mu\mathcal{V}_+\mathcal{V}_+^{-1} &= (e_1^1)^{-1}\partial_\mu e_1^1 \hat{E}_1^1 + (e_2^2)^{-1}\partial_\mu e_2^2 \hat{E}_2^2 + e^{-1}\partial_\mu e \hat{E}_3^3 \\ &\quad - e_1^1(e_2^2)^{-1}\partial_\mu B \hat{E}_2^1 + e^{-1}\left(e_1^1(\partial_\mu\psi_1 - B\partial_\mu\psi_2)\hat{E}_3^1 + e_2^2\partial_\mu\psi_2 \hat{E}_3^2\right) \\ &\quad + \partial_\mu \tilde{e}_{\tilde{a}}^{\tilde{i}} \tilde{e}_{\tilde{i}}^{\tilde{b}} \hat{E}_{\tilde{b}}^{\tilde{a}} \end{aligned} \quad (57)$$

³ Here we use dotted numbers for the flat local Lorentz (though Euclidean here) indices $a' = \dot{1}, \dot{2}$, to distinguish them from the curved tangent space indices $i' = 1, 2$ for the reduced dimensions.

$$\left(e_{a'}^{i'} = \begin{pmatrix} e_1^1 & -e_1^1 B \\ 0 & e_2^2 \end{pmatrix} \right), \text{ and}$$

$$\begin{aligned} \mathcal{V}_+(\partial_\mu \mathcal{V}_- \mathcal{V}_-^{-1}) \mathcal{V}_+^{-1} &= e_{a'}^{i'} \overset{\circ}{f}_I^A \partial_\mu A_{i'}^I E_A^{*a'} \\ &+ e^{-1} \overset{\circ}{f}_A^I \left(\partial_\mu \varphi_I - \frac{1}{2} C_{JKI} \epsilon^{i'j'} A_{i'}^J \partial_\mu A_{j'}^K \right) E_3^A \\ &+ e^{-1} e_{a'}^{i'} \left(2(A_{i'}^I \partial_\mu \varphi_I - \partial_\mu A_{i'}^I \varphi_I) - \frac{2}{3} C_{JKI} \epsilon^{j'k'} A_{i'}^J A_{j'}^K \partial_\mu A_{k'}^I \right) \hat{E}^{a'}_3, \end{aligned} \quad (58)$$

where

$$\overset{\circ}{f}_I^A = (\exp((\log \tilde{\mathbf{e}}^{-1})_{\tilde{a}}^{\tilde{b}} \bar{T}_{\tilde{b}}^{\tilde{a}}))_I^A, \quad (59)$$

$$\overset{\circ}{f}_A^I = (\exp((\log \tilde{\mathbf{e}}^{-1})_{\tilde{a}}^{\tilde{b}} T_{\tilde{b}}^{\tilde{a}}))_A^I \quad (60)$$

are respectively the $\bar{\mathbf{6}}$ and $\mathbf{6}$ representation matrices of the $\widetilde{SL}(3, \mathbb{R})$ group element $\tilde{\mathbf{e}}^{-1}$ (54).

Plugging (57)(58) into (47), $\frac{1}{2}(\partial_\mu \mathcal{V} \mathcal{V}^{-1} + \tau(\partial_\mu \mathcal{V} \mathcal{V}^{-1}))$ projects out the \mathbf{H} piece of $\partial_\mu \mathcal{V} \mathcal{V}^{-1}$, leaving only the \mathbf{K} piece. This amounts to the replacements

$$\begin{aligned} \hat{E}^i_j &\longrightarrow \frac{1}{2}(\hat{E}^i_j + \hat{E}^j_i), \\ \hat{\tilde{E}}^{\tilde{a}}_{\tilde{b}} &\longrightarrow \frac{1}{2}(\hat{\tilde{E}}^{\tilde{a}}_{\tilde{b}} + \hat{\tilde{E}}^{\tilde{b}}_{\tilde{a}}), \\ E^I_i &\longrightarrow \frac{1}{2}(E^I_i + E^{*I}_i), \\ E^{*I}_I &\longrightarrow \frac{1}{2}(E^I_i + E^{*I}_I) \end{aligned} \quad (61)$$

in $\partial_\mu \mathcal{V} \mathcal{V}^{-1}$. Thus, using the invariant bilinear form computed in the adjoint representation normalized by twice the dual Coxeter number $2h_{F_4}^\vee = 18$:⁴

$$\begin{aligned} \frac{1}{18} \text{Tr} E_b^a E_d^c &= \delta_b^c \delta_d^a \quad (a, b, c, d = 1, 2, 3), \\ \frac{1}{18} \text{Tr} \tilde{E}_{\tilde{b}}^{\tilde{a}} \tilde{E}_{\tilde{d}}^{\tilde{c}} &= 2 \delta_{\tilde{b}}^{\tilde{c}} \delta_{\tilde{d}}^{\tilde{a}} \quad (\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d} = 1, 2, 3), \\ \frac{1}{18} \text{Tr} E_a^A E_B^{*b} &= 4 \delta_a^b \delta_B^A \quad (a, b = 1, 2, 3; \quad A, B = 1, \dots, 6), \\ \text{otherwise} &= 0, \end{aligned} \quad (62)$$

⁴ It is simpler to use $E_b^a, \tilde{E}_{\tilde{b}}^{\tilde{a}}$ than to use hatted generators to compute traces, where $\hat{E}_b^a = E_b^a - \frac{1}{3} \delta_b^a (E_1^1 + E_2^2 + E_3^3)$ and similarly for $\hat{\tilde{E}}_{\tilde{b}}^{\tilde{a}}$

we obtain

$$\begin{aligned}
\frac{1}{72} \text{Tr} \partial_\mu \mathcal{M}^{-1} \partial^\mu \mathcal{M} = & \frac{1}{4} \partial_\mu g^{ij} \partial_\mu g_{ij} - e^{-2} \partial_\mu e \partial^\mu e + \frac{1}{2} \partial_\mu \tilde{g}^{ij} \partial_\mu \tilde{g}_{ij} - 2g^{ij} \overset{\circ}{a}_{IJ} \partial_\mu A_i^I \partial^\mu A_j^J \\
& - 2e^{-2} \overset{\circ}{a}^{IJ} \left(\partial_\mu \varphi_I - \frac{1}{2} C_{KLI} \epsilon^{kl} A_k^K \partial_\mu A_l^L \right) \\
& \cdot \left(\partial^\mu \varphi_J - \frac{1}{2} C_{K'L'J} \epsilon^{k'l'} A_{k'}^{K'} \partial_\mu A_{l'}^{L'} \right) \\
& - \frac{1}{2} e^{-2} g^{ij} \left(\partial_\mu \psi_i - 2(\varphi_I \partial_\mu A_i^I - \partial_\mu \varphi A_i^I) - \frac{2}{3} C_{KLI} \epsilon^{kl} A_k^K \partial_\mu A_l^L A_i^I \right) \\
& \cdot \left(\partial^\mu \psi_j - 2(\varphi_I \partial_\mu A_j^J - \partial_\mu \varphi A_j^J) - \frac{2}{3} C_{K'L'J} \epsilon^{k'l'} A_{k'}^{K'} \partial_\mu A_{l'}^{L'} A_j^J \right).
\end{aligned} \tag{63}$$

This final form of the sigma model coincides with $2E^{-1}$ times the dimensionally reduced Lagrangian (7) obtained in the previous section with the rescalings

$$A_i^I \rightarrow \frac{A_i^I}{\sqrt{2}}, \quad \varphi_I \rightarrow \frac{\varphi_I}{\sqrt{2}}, \quad \psi_i \rightarrow 2\psi_i, \quad C_{IJK} \rightarrow \frac{4}{\sqrt{3}} C_{IJK}. \tag{64}$$

This complete the direct proof of the equivalence of the dimensionally reduced Lagrangian of the magical supergravity to the $F_{4(+4)}/(USp(6)) \times SU(2)$ nonlinear sigma model.

IV. CONCLUSIONS AND DISCUSSION: OTHER MAGICAL SUPERGRAVITIES

In this letter we have shown the direct relationship between the (bosonic part of the) simplest of the four magical theories reduced to three dimensions and the $F_{4(+4)}/(USp(6)) \times SU(2)$ coset sigma model. As we mentioned in Introduction, these relations will be used to generate various new supergravity solutions by applying $F_{4(+4)}$ transformations to some known solutions of this magical supergravity.

We can give some Lie algebraic characterizations to various geometrical quantities defined in the supergravity Lagrangian:

- C_{IJK} 's are the structure constants of the commutation relations between generators both belonging to $(\mathbf{3}, \bar{\mathbf{6}})$. In particular $I = 1, \dots, 6$ are the indices for a symmetric tensor representation $\bar{\mathbf{6}}$ of the $SL(3, \mathbb{R})$, which is the numerator group of the scalar coset $SL(3, \mathbb{R})/SO(3)$ already existing in five dimensions.
- $\overset{\circ}{a}^{IJ}$ and $\overset{\circ}{a}_{IJ}$ are nothing but the $\mathbf{6}$ and $\bar{\mathbf{6}}$ representation matrices of the metric of the reduced two dimensions viewed as an $SL(3, \mathbb{R})$ group element.

We note that the structures we found here are very similar to the dimensionally reduced eleven-dimensional supergravity or the $D = 5$ minimal supergravity to three dimensions [14, 23, 24], whose sigma models are respectively $E_{8(+8)}/SO(16)$ and $G_{2(+2)}/SO(4)$.

In all the magical supergravity theories, the number of the original scalars ($= n$) is always one less than the number of the abelian gauge fields. In the simplest magical case considered in this letter, this is the number of the dimension of the *symmetric tensor* representation, which is 6. In fact, for the other three magical cases, we can also find representations of the numerator group of the coset whose dimensions are *precisely* one more than the dimensions of the coset of the respective theories [22]:

- $J_3^{\mathbb{C}}$ magical:

$$E_{6(+2)} \supset SL(3, \mathbb{R}) \times SL(3, \mathbb{C}) = SL(3, \mathbb{R}) \times (SL(3, \mathbb{R}) \times SL(3, \mathbb{R})) \quad (65)$$

$$\mathbf{78} = (\mathbf{8}, (\mathbf{1}, \mathbf{1})) \oplus (\mathbf{3}, (\bar{\mathbf{3}}, \bar{\mathbf{3}})) \oplus (\bar{\mathbf{3}}, (\mathbf{3}, \mathbf{3})) \oplus (\mathbf{1}, (\mathbf{8}, \mathbf{1})) \oplus (\mathbf{1}, (\mathbf{1}, \mathbf{8})). \quad (66)$$

The dimension of the five-dimensional scalar coset is

$$\dim \frac{SL(3, \mathbb{C})}{SU(3)} = 8, \quad (67)$$

so the index I runs from 1 to 9. This agrees with the fact that the *direct product* representation $(\mathbf{3}, \mathbf{3})$ or $(\bar{\mathbf{3}}, \bar{\mathbf{3}})$ is nine-dimensional.

- $J_3^{\mathbb{H}}$ magical:

$$E_{7(-5)} \supset SL(3, \mathbb{R}) \times SU^*(6) \quad (68)$$

$$\mathbf{133} = (\mathbf{8}, \mathbf{1}) \oplus (\mathbf{3}, \bar{\mathbf{15}}) \oplus (\bar{\mathbf{3}}, \mathbf{15}) \oplus (\mathbf{1}, \mathbf{35}). \quad (69)$$

The dimension of the coset is

$$\dim \frac{SU^*(6)}{USp(6)} = 14. \quad (70)$$

In this case the relevant representations are the *rank-2 antisymmetric tensor* representations, which are $\mathbf{15}$ and $\bar{\mathbf{15}}$.

- $J_3^{\mathbb{O}}$ magical:

$$E_{8(-24)} \supset SL(3, \mathbb{R}) \times E_{6(-26)} \quad (71)$$

$$\mathbf{133} = (\mathbf{8}, \mathbf{1}) \oplus (\mathbf{3}, \bar{\mathbf{27}}) \oplus (\bar{\mathbf{3}}, \mathbf{27}) \oplus (\mathbf{1}, \mathbf{78}). \quad (72)$$

In this case

$$\dim \frac{E_{6(-26)}}{F_4} = 26. \quad (73)$$

This also agrees with the existence of the *fundamental* **27** and $\overline{\mathbf{27}}$ representations of E_6 with the above decomposition of $E_{8(-24)}$.

In view of this common structure of decompositions (known as the decomposition of the quasi-conformal algebra of the Jordan algebra in terms of the super-Ehlers' algebra [22]), we expect the same characterization for C_{IJK} or $\overset{\circ}{a}^{IJ}$ and $\overset{\circ}{a}_{IJ}$ will be possible for the other three magical supergravity theories. To show this the realizations worked out in [20] will be useful. Work along this line is in progress.

Acknowledgments

We would like to thank A. Ishibashi, H. Kodama and S. Tomizawa for discussions. A conversation had with H. Nicolai some time ago has been also useful, for which he is also acknowledged. The work of S. M. is supported by Grant-in-Aid for Scientific Research (C) #25400285, (C) #16K05337 and (A) #26247042 from The Ministry of Education, Culture, Sports, Science and Technology of Japan.

-
- [1] E. Cremmer and B. Julia, Phys. Lett. B **80** (1978) 48; Nucl. Phys. B **159** (1979) 141.
 - [2] E. Cremmer, B. Julia and J. Scherk, Phys. Lett. B **76** (1978) 409.
 - [3] B. Julia, Conf. Proc. C **8006162** (1980) 331.
 - [4] A. Keurentjes, Nucl. Phys. B **658** (2003) 303 [hep-th/0210178].
 - [5] C. M. Hull and P. K. Townsend, Nucl. Phys. B **438** (1995) 109 [hep-th/9410167].
 - [6] N. Marcus and J. H. Schwarz, Nucl. Phys. B **228** (1983) 145.
 - [7] H. Nicolai, Phys. Lett. B **194** (1987) 402.
 - [8] B. Julia, Kac-Moody Symmetry of Gravitation and Supergravity Theories, AMS-SIAM Summer Seminar on Applications of Group Theory in Physics and Mathematics, Chicago (1982).
 - [9] R. W. Gebert and H. Nicolai, E10 for Beginners, Gursey Memorial Conference I: On Strings and Symmetries, Istanbul (1994).
 - [10] R. Geroch, J. Math. Phys. **13** (1972) 394.

- [11] P. Breitenlohner and D. Maison, Ann. Inst. Henri Poincaré, 46 (1987) 216.
- [12] J. Maharana and J. H. Schwarz, Nucl. Phys. B **390** (1993) 3 [hep-th/9207016].
- [13] A. Sen, Int. J. Mod. Phys. A **9** (1994) 3707 [hep-th/9402002].
- [14] S. Mizoguchi and N. Ohta, Phys. Lett. B **441** (1998) 123 [hep-th/9807111].
- [15] M. Gunaydin, G. Sierra and P. K. Townsend, Nucl. Phys. B **242** (1984) 244.
- [16] M. Gunaydin, G. Sierra and P. K. Townsend, Phys. Lett. B **133** (1983) 72.
- [17] P. Karndumri, JHEP **1208** (2012) 007 doi:10.1007/JHEP08(2012)007 [arXiv:1206.2150 [hep-th]].
- [18] P. Karndumri, JHEP **1512** (2015) 153 doi:10.1007/JHEP12(2015)153 [arXiv:1509.07431 [hep-th]].
- [19] S. Mizoguchi and S. Tomizawa, Phys. Rev. D **84** (2011) 104009 [arXiv:1106.3165 [hep-th]].
S. Tomizawa and S. Mizoguchi, Phys. Rev. D **87** (2013) no.2, 024027 [arXiv:1210.6723 [hep-th]].
- [20] M. Gunaydin, K. Koepsell and H. Nicolai, Commun. Math. Phys. **221** (2001) 57 [hep-th/0008063].
- [21] M. Gunaydin and O. Pavlyk, JHEP **0501** (2005) 019 [hep-th/0409272]
- [22] S. Ferrara, A. Marrani and B. Zumino, J. Phys. A **46** (2013) 065402 [arXiv:1208.0347 [math-ph]].
- [23] S. Mizoguchi, Nucl. Phys. B **528** (1998) 238 [hep-th/9703160].
- [24] S. Mizoguchi and G. Schroder, Class. Quant. Grav. **17** (2000) 835 [hep-th/9909150].